The Plancherel–Hurwitz measure: random partitions meet random maps at high genus

Random integer partitions are particularly useful for studying random permutations. Famously, random partitions of n under the Plancherel measure induced by the RSK algorithm (which maps permutations bijectively to pairs of Young tableaux of the same shape) determine the lengths of the monotone subsequences of uniform random permutations of the numbers $(1, \dots, n)$. A natural generalisation of this measure relating to factorisations of the identity on S_n by ℓ transpositions is induced not by a bijection but by a purely algebraic relation. Namely, by Frobenius' formula, the unconnected classical Hurwitz numbers $H_{n,\ell}$ satisfy

$$H_{n,\ell} := \#\{(\tau_1, \tau_2, \dots, \tau_\ell) \in (S_n)^\ell | \tau_1\tau_2 \cdots \tau_\ell = \mathrm{id}\} = \sum_{\lambda \vdash n} {f_\lambda}^2 {C_\lambda}^\ell \tag{1}$$

where each τ_i denotes a transposition on S_n and f_λ and C_λ are Young diagrammatic quantities, the number of Young tableaux of shape λ and the sum of the contents of λ. For even ℓ, we define the *Plancherel–Hurwitz measure* on partitions of n by

$$\mathbb{P}_{n,\ell}(\lambda) = \frac{1}{H_{n,\ell} n!} {f_\lambda}^2 {C_\lambda}^\ell. \tag{2}$$

This measure was indirectly studied by Diaconis and Shahshahani in work on random shuffling [1], and was related to tau-funtions of the Toda lattice integrable hierarchy by Okounkov [2]. At $\ell = 0$, it reduces to the classical Plancherel measure.

We study the asymptotic behaviour of these measures in a new regime where the number of transpositions ℓ grows linearly with the order of the group n, and find a novel two-fold limit shape, where the first part is much larger than the others (see Figure 1). As a consequence, we obtain asymptotic estimates for the unconnected Hurwitz numbers $H_{n,\ell}$ in this regime, which can be interpreted as the return probability of the transposition random walk on S_n after ℓ steps.

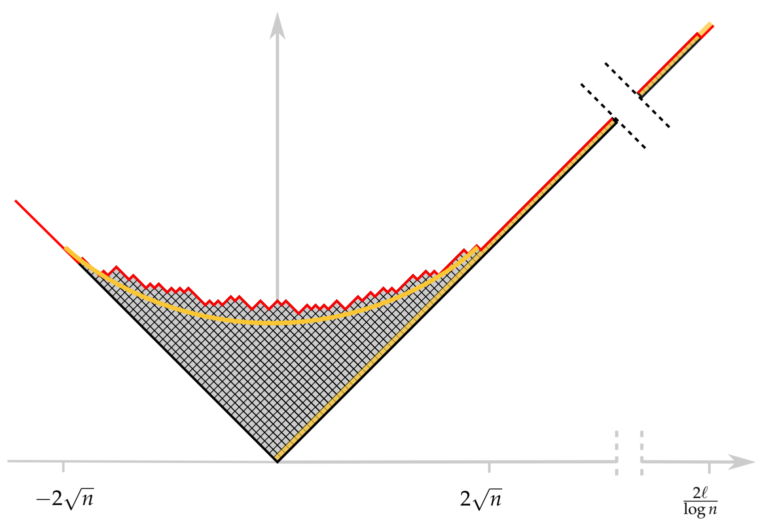

Figure 1: A partition of $n = 2500$ sampled under the Plancherel–Hurwitz measure at high-genus, and its limit shape. The first part grows with $\frac{n}{\log n}$ and the rest of the partition has the Plancherel measure limit shape. [3]

However, this work was motivated by another interpretation: the transposition factorisations counted by $H_{n,\ell}$ are in bijection with a particular family of discretised not necessarily connected surfaces called *pure Hurwitz maps* introduced by Bousquet-Mélou and Schaeffer [4] (see figure 2).

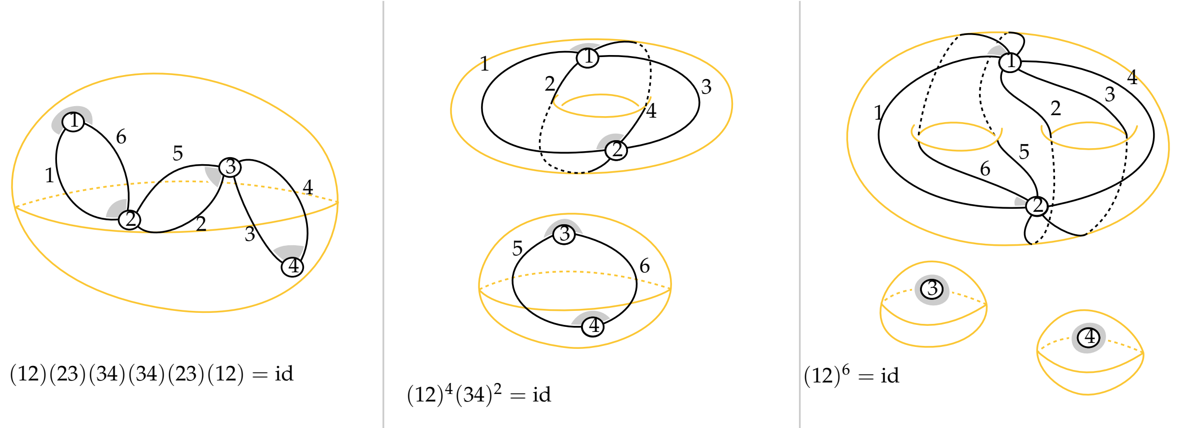

Figure 2: Unconnected pure Hurwitz maps with Euler characteristic $\chi = 2$ corresponding to factorisations of the identity by 6 transpositions on S_4. Each transposition gives the labels of two vertices which are connected by an edge, the order of the transpositions determines the cyclic ordering of the edges around vertices.

In the corresponding map model, ℓ growing linearly with n is a *high genus* regime. Such regimes represent an important new frontier in the combinatorics of discrete surfaces, where established tools such as matrix model generating functions fail and where the approximate asymptotic enumeration results found for connected maps by Budzinski and Louf [5] were only possible using a combination of probabilistic and algebraic methods. We use our asymptotic estimate for $H_{n,\ell}$ to study the connectedness of uniform random Hurwitz maps, and find that they contain a large number of isolated vertices. We discuss how we hope to extend our approach to find asymptotic estimates for connected Hurwitz numbers at high genus, and to interpret the statistics of the Plancherel–Hurwitz measure in terms of transposition factorisations.

Based on joint work with Guillaume Chapuy and Baptiste Louf [3].

References

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